Descriptive Set Theory HW 2

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Problem 1. Let (X, d) be a metric with $d \leq 1$. For sequence $(K_n)_{n\in\mathbb{N}} \subseteq \mathcal{K}(X) - \{\varnothing\}$ and nonempty $K \in \mathcal{K}(X)$, show the following:

1. If $\delta(K, K_n) \to 0$, then $K \subseteq \underline{\mathrm{T}\lim_n} K_n$;
2. If $\delta(K_n, K) \to 0$, then $K \supseteq \overline{\mathrm{T}\lim_n} K_n$;

In particular, $d_H(K_n, K) \to 0 \Rightarrow K = \mathrm{T}\lim_n K_n$. Show that the converse may fail.

Solution.

1. Assume that $\delta(K, K_n) \to 0$, and fix $a \in K$. To show $a \in \underline{\mathrm{T}\lim_n} K_n$, it's enough to find a sequence $(x_n) \in \prod K_n$ that converges to a. Towards that end, observe that $d(a, K_n) \leq \delta(K, K_n) \to 0$. So, we may construct $(N_i) \in \omega^\omega$ by choosing $N_{i+1} > N_i$ to satisfy $d(a, K_n) < 2^{-i-1}$ for $n \geq N_{i+1}$. Now, define $(x_n) \in \prod K_n$ as follows: if $n < N_0$, pick $x_n \in K_n$ to be arbitrary. Otherwise, if $N_i \leq n < N_{i+1}$, pick $x_n \in K_n$ to witness that $d(a, K_n) < 2^{-i}$. By construction, $d(x_n, a)$ is monotonically decreasing to 0 as $n \to \infty$. So $(x_n) \to a$ as desired.

2. Assume that $\delta(K_n, K) \to 0$ and fix $a \in \overline{\mathrm{T}\lim_n} K_n$. We may fix a sequence $(x_n) \in \prod K_n$ such that $(x_{n_i}) \to a$ for some subsequence. Our goal is to find a sequence $(a_k) \in K^\omega$ such that $(a_k) \to a$. Since K is closed, we'd win. To do this, we construct sequences $(N_k) \in \omega^\omega$ and $(a_k) \in K^\omega$ as follows: given a_k and N_k, let $m > N_k$ be such that $\delta(K_l, K) < 2^{-k-1}$ if $l \geq m$. Let $i_{k+1} > 0$ be least such that $n_{i_{k+1}} \geq m$. Then, choose $a_{k+1} \in K$ to witness $d(x_{n_{i_{k+1}}}, a_{k+1}) < 2^{-k-1}$. We can do this because $d(x_{n_{i_{k+1}}}, K) \leq \delta(K_{n_{i_{k+1}}}, K) < 2^{-k-1}$. Then, complete the construction by setting $N_{k+1} = n_{i_{k+1}} + 1$.

   To see that this works, notice that the sequence $(x_{n_{i_k}}) \to a$ as $(x_{n_i}) \to a$. By construction, the a_k's get arbitrarily close to the $x_{n_{i_k}}$'s. It follows that $(a_k) \to a$ as well.

To see that the converse fails, we observe Example 3.12(b) in Anush's notes. If $X = \mathbb{R}$ and $K_n = [0,1] \cup [n, n+1]$, then $\mathrm{T}\lim_n K_n = [0,1]$ but the sequence doesn't converge in the Hausdorff metric. $\star$

Problem 2. Let (X, d) be a metric space with $d \leq 1$. Then $x \mapsto \{x\}$ is an isometric embedding of X into $\mathcal{K}(X)$.

Solution. The map is obviously injective. To see this is an isometry, notice that $\delta(\{x\}, \{y\}) = d(x, \{y\}) = d(x, y)$, and so $d(x, y) = d_H(\{x\}, \{y\})$. $\star$

Problem 3. Let (X, d) be a metric space with $d \leq 1$ and assume $K_n \to K$. Then any sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in K_n$ has a subsequence converging to a point in K.

Solution. We construct a sequence (y_i) of elements of K as follows: for each $k > 0$, find $N_k > N_{k-1}$ such that for each $m \geq N_k$, $d_H(K_m, K) < 2^{-k}$. This implies that $\delta(K_m, K) < 2^{-k}$, and so in particular, we have that $d(x_{N_k}, K) < 2^{-k}$. So, fix $y_k \in K$ such that $d(x_{N_k}, y_k) < 2^{-k}$. Since K is compact, there's a convergent subsequence $(y_{k_i}) \to y$. We claim that $(x_{N_{k_i}})_i \to y$. Given $k > 0$, find $i > 0$ such that $k_i > k$. Indeed, observe that

$$d(x_{N_{k_i}}, y) \leq d(x_{N_{k_i}}, y_{k_i}) + d(y_{k_i}, y).$$

As $i \to \infty$, we have not only that $d(y_{k_i}, y) \to 0$ but also $d(x_{N_{k_i}}, y_{k_i}) \to 0$ by construction. The result follows. $\star$

Problem 4. Let X be metrizable.

1. The relation $x \in K$ is closed in $X \times \mathcal{K}(X)$.
2. The relation $K \subseteq L$ is closed in $\mathcal{K}(X)^2$.
3. The relation $K \cap L \neq \varnothing$ is closed in $\mathcal{K}(X)^2$.

Solution.

Fix a compatible metric $d \leq 1$ on X.

1. Assume that $x_n \to x$ and $K_n \to K$, where $x_n \in K_n$ for each n. By the previous problem, there's a subsequence of x_n converging to a point in K. But, then this subsequence must converge to x, implying $x \in K$.
2. Assume that $K_n \to K$ and $L_n \to L$ where $K_n \subseteq L_n$ for each n. So, it follows that $\overline{\mathrm{T}\lim_n} K_n \subseteq \overline{\mathrm{T}\lim_n} L_n$. By one of the previous HW problems, we get that $K = \overline{\mathrm{T}\lim_n} K_n$ and $L = \overline{\mathrm{T}\lim_n} L_n$. But then $K \subseteq L$ and we win.

3. We show that $K \cap L = \varnothing$ is an open relation. To see this, first observe that metric spaces are normal. Given disjoint K and L, we find split K and L by disjoint open sets U and V. Then, for any $F \in \langle U; \rangle_{\mathcal{K}}$ and for any $G \in \langle V; \rangle_{\mathcal{K}}$, we have that $F \cap G = \varnothing$.

$\star$

Problem 5. Let X be a topological space.

1. If X is nonempty perfect, then so is $\mathcal{K}(X) - \{\varnothing\}$.
2. If X is compact metrizable, then $C(X)$ is perfect, where $C(X) = C(X, \mathbb{R})$.

Solution.

1. Fix a nonempty compact $K \subseteq X$, and assume instead that $\{K\}$ were open. So, it follows that

$$\{K\} = \{F \colon F \subseteq U_1, F \cap U_2 \neq \varnothing, \dots, F \cap U_k \neq \varnothing\},$$

for some open $U_1, \dots, U_k$ with $U_1 \supseteq U_2, \dots, U_k$. For the sake of sanity, denote the RHS by $\mathcal{U}$. If there is a x_1 such that $x_1 \in U_1 - K$, then it follows that $K \cup \{x_1\}$ and K are both in $\mathcal{U}$, contradicting that $\mathcal{U} = \{K\}$. So we may assume that $K = U_1$. Since X is perfect, we may also assume that $k \geq 2$.

Now, define a subset $I \subseteq \{2, \dots, k\}$ to be **good** if $\bigcap_{i \in I} U_i \neq \varnothing$. Let $\mathcal{G}$ be the set of all good subsets that are maximal wrt the subset ordering. For each $I \in \mathcal{G}$, choose $x_I \in \bigcap_{i \in I} U_i$. First, observe that $\{x_I\}_{I \in \mathcal{G}} \in \mathcal{U}$, as for each $a \in \{2, \dots, k\}$, we can find an $I \in \mathcal{G}$ containing a. We win after showing that $\{x_I\}_{I \in \mathcal{G}} \subsetneq K$. To see this, fix any $I \in \mathcal{G}$. Since $\bigcap_{i \in I} U_i \neq \varnothing$ and X is perfect, there's a $y \in \bigcap_{i \in I} U_i$ such that $x_I \neq y$. Now, if $y = x_J$ for some $J \in \mathcal{G}$, then $y \in \bigcap_{i \in I \cup J} U_i$. Since I and J are both maximal wrt the subset ordering, it follows that $I \cup J \subseteq I, J$, implying that $I = J$. This implies that $y \neq x_J$ for any $J \in \mathcal{G}$, yielding that $K - \{x_I\}_{I \in \mathcal{G}}$ is nonempty. The result follows.

2. To see that $C(X)$ is perfect, fix $\varepsilon > 0$ and $f \in C(X)$. Define $g(x) = f(x) + \frac{\varepsilon}{2}$ for all $x \in X$. Then, $g \in B(f, \varepsilon)$, and so $|B(f, \varepsilon)| \geq 2$. It follows that $C(X)$ is perfect.

$\star$

Problem 6. Show that any nonempty perfect compact Hausdorff space X has cardinality at least continuum by constructing an injection from the Cantor space into X.

Solution. We first begin with a claim:

Claim 1. For each nonempty open $U \subseteq X$, there's a nonempty open V such that $\overline{V} \subseteq U$.

Proof. Fix $x \in U$. Since X is a compact Hausdorff space, X is regular. So, we may find an open nhbd V of x and an open $W \supseteq U^c$ such that $V \cap W = \varnothing$. Since W^c is closed, it follows that $\overline{V} = \bigcap\{F \text{ closed} : F \supseteq V\} \subseteq W^c \subseteq U$, as desired. □

Now, we construct a Cantor scheme $(U_s)_{s \in 2^{<\omega}}$ of nonempty open sets in the following way:

1. Set $U_\varnothing = X$.

2. Given a nonempty U_s, we may fix distinct $x, y \in U_s$, because X is perfect. Since X is Hausdorff, we may **choose** disjoint open nhbds $V_0, V_1 \subseteq U_s$ of x and y respectively. Applying the above claim twice, we may therefore **choose** appropriate disjoint open nhbds such that $\overline{U_{s^\frown 0}} \subseteq V_0 \subseteq U_s$ and $\overline{U_{s^\frown 1}} \subseteq V_1 \subseteq U_s$. This completes the construction.

   (Remark: I think just dependent choice was used here, as our construction is done in countably many steps, and we made our choice of open sets depending on the open set U_s we were given.)

Since X is compact, it follows that $\bigcap_n U_{x|n} = \bigcap_n \overline{U_{x|n}} \neq \varnothing$ for $x \in 2^\omega$. Further, we have by construction that the $\bigcap_n U_{x|n}$ are disjoint for different elements of Cantor Space. This induces an injection from 2^ω into X as desired.

(Remark: Since X wasn't a complete metric space, I don't think we know that $\bigcap_n U_{x|n}$ is a singleton. In which case, it seems we would have to choose an element in $\bigcap_n U_{x|n}$ for each $x \in 2^\omega$, which involves more than just the dependent choice used above.) ⋆

Problem 7. Let X be a nonempty perfect Polish space and let Q be a countable dense subset of X. Show that Q is F_σ but not G_δ. Conclude that $\mathbb{Q}$ is not Polish in the relative topology $\mathbb{R}$.

Solution. Q is F_σ because Q is countable, $\{x\}$ is closed, and $Q = \bigcup_{x \in Q} \{x\}$. We claim that Q is a perfect subspace. Indeed, given your favorite open $U \subseteq X$, we may find distinct $x, y \in U$, because X is perfect. Since X is Hausdorff, we may find disjoint open nhbds $U_0, U_1 \subseteq U$ of x, y respectively. Since Q is dense, it follows that $U_0 \cap Q$ and $U_1 \cap Q$ are nonempty. Since these are disjoint, it follows that $U \cap Q$ contains at least 2 elements, implying that Q is a perfect subspace.

To see why this is enough, observe that if Q were also a G_δ subset of X, then Q with the relative topology would be a nonempty perfect Polish space. Hence, Q has cardinality continuum, contradicting that Q is countable. The result follows for $\mathbb{Q}$ because it's a countable dense subset of $\mathbb{R}$.

Another argument would be that Q would be a comeager set, as it would be a dense G_δ. But, Q is meager as it's the countable union of singletons. This would imply that X is meager, which contradicts the Baire Category Theorem.

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Problem 8. Show that the perfect kernel of a Polish space X is the largest perfect subset of X, i.e. it contains all other perfect subsets.

Solution. Recall, X may be uniquely written to have the form $X = P \cup C$, where the perfect kernel P is the set of all condensation points of X, and C is countable.

Now, assume that Q is a perfect subset of X. First, note that Q is in fact Polish, as it's closed by definition, and X is a Polish space by hypothesis. Our goal is to show that $Q \subseteq P$, so fix $x \in Q$ and an open nhbd U of x. We must show that U is uncountable. Since Q is perfect, it follows that $U \cap Q$ is a perfect subspace of Q, as open subsets of $U \cap Q$ are also open in Q. This implies $U \cap Q$ is a nonempty perfect Polish space, as Q is Polish and $U \cap Q \subseteq Q$ is an open subset of a Polish space. Well, then we win, as $U \cap Q$ will be uncountable, implying that $x \in P$, as desired. $\star$